The Functional Integration and the Two-Point Correlation Functions of the Trapped Bose Gas

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Abstract

A quantum field-theoretical model, which describes spatially non-homogeneous repulsive Bose gas in an external harmonic potential is considered. Two-point thermal correlation functions of the Bose gas are calculated in the framework of the functional integration approach. Successive integration over the “high-energy” functional variables first and then over the “low-energy” ones is used. The effective action functional for the low-energy variables is obtained in one loop approximation. The functional integral representations for the correlation functions are estimated by means of the stationary phase approximation. A power-law asymptotical behaviour of the correlators of the one-dimensional Bose gas is demonstrated in the limit, when the temperature is going to zero, while the volume occupied by the non-homogeneous Bose gas infinitely increases. The power-law behaviour is governed by the critical exponent dependent on the spatial arguments.

1 Introduction

Experimental realization of Bose condensation in vapours of alcali metals confined in the magneto-optical traps stimulated a considerable interest to the theory of the Bose gas [1,2]. In particular, Bose condensation in the systems, which are effectively two-dimensional or quasi one-dimensional, became a subject of experimental and theoretical investigations. For more details one should be referred to [1,2]. The field models, which describe the Bose particles with delta-like interparticle coupling confined by an external harmonic potential, provide a reliable background for theoretical description of experimental situations [1,2]. For a translationally invariant case, the field models in question correspond to a quantum nonlinear Schrödinger equation which allows to obtain closed expressions for the correlation functions in the one-dimensional case [3].

Some of the results of the papers [4–6] devoted to the correlation functions of the weakly repulsive Bose gas confined by an external harmonic potential are reported below. Since there are no exact solutions in the case of an external potential, the functional integration approach (see [7–13] as a list, though incomplete, of appropriate refs.) is used in [4–6] for investigation of the two-point thermal correlation functions. It will be demonstrated below that the presence of the external potential results in a modification of the asymptotical behaviour of the correlation functions in comparison to a translationally invariant case.

The paper is organized as follows. Section 1 has an introductory character. A description of the one-dimensional model of non-relativistic Bose field in question, as well as a summary of the functional integration approach, are given in Section 2. Successive integration over the high-energy over-condensate excitations first and then over the variables, which correspond to the low-energy quasi-particles, is used in the given paper for a derivation of one loop effective action for the low excited quasi-particles. The method of stationary phase is used in Section 3 for approximate investigation of the functional integrals, which express the two-point thermal correlation functions. Specifically, the asymptotical approach to estimation of the correlators, which is discussed in the present paper, was proposed in [14]. It is clear after [4–6] that the method [14] admits a generalization for the spatially non-homogeneous Bose gas in the external potential as well. The asymptotics of the two-point correlation functions of the non-homogeneous one-dimensional Bose gas are obtained in Section 4. A short discussion in Section 5 concludes the paper.

2 The effective action and the Thomas–Fermi approximation

2.1 The partition function

Let us consider one-dimensional repulsive Bose gas on the real axis $\mathbb{R} \ni x$ confined by an external potential $V(x)$. We represent its partition function Z in the form of the functional integral [7–13]:

$$Z = \int \mathrm{e}^{S[\psi,\bar{\psi}]} \mathcal{D}\psi \mathcal{D}\bar{\psi}\,, \tag{1}$$

where $S[\psi,\bar{\psi}]$ is the action functional:

$$\begin{aligned} S[\psi,\bar{\psi}] &= \int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \Big\{ \bar{\psi}(x,\tau)\Big(\frac{\partial}{\partial\tau} - \mathcal{H}\Big)\psi(x,\tau) - \\ &\qquad -\frac{g}{2}\,\bar{\psi}(x,\tau)\bar{\psi}(x,\tau)\psi(x,\tau)\psi(x,\tau)\Big\}\,, \end{aligned} \tag{2}$$

and $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ is the functional integration measure. Other notations in (1), (2) are: $\mathcal{H}$ is the "single-particle" Hamiltonian,

$$\mathcal{H} \equiv \frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu + V(x)\,, \tag{3}$$

m is the mass of the Bose particles, μ is the chemical potential, g is the coupling constant corresponding to the weak repulsion (i.e., $g > 0$), and the external confining potential is $V(x) \equiv \frac{1}{2}m\Omega^2 x^2$. The domain of the functional integration in (1) is given by the space of the complex-valued functions $\bar{\psi}(x,\tau)$, $\psi(x,\tau)$ depending on $x \in \mathbb{R}$ and $\tau \in [0,\beta]$. With regard to x, the functions $\bar{\psi}(x,\tau)$, $\psi(x,\tau)$ belong to the space of quadratically integrable functions $L_2(\mathbb{R})$, while they are finite and periodic with the period $\beta = (k_B T)^{-1}$ with regard to the imaginary time τ (k_B is the Boltzmann constant, and T is an absolute temperature).

At sufficiently low temperatures each of the variables $\bar{\psi}(x,\tau)$, $\psi(x,\tau)$ is given by two constituents:

$$\psi(x,\tau) = \psi_o(x,\tau) + \psi_e(x,\tau)\,, \quad \bar{\psi}(x,\tau) = \bar{\psi}_o(x,\tau) + \bar{\psi}_e(x,\tau)\,, \tag{4}$$

where $\bar{\psi}_o(x,\tau)$, $\psi_o(x,\tau)$ correspond to *quasi-condensate* (a true Bose condensate does not exist in one-dimensional systems [11]), while $\bar{\psi}_e(x,\tau)$, $\psi_e(x,\tau)$ correspond to the high-energy thermal (i.e., over-condensate) excitations. In the exactly solvable case, the existence of the quasi-condensate implies that a non-trivial ground state exists [3]. Let us require the variables (4) to be orthogonal in the following sense:

$$\int \psi_o(x,\tau)\bar{\psi}_e(x,\tau)\,\mathrm{d}x = \int \bar{\psi}_o(x,\tau)\psi_e(x,\tau)\,\mathrm{d}x = 0\,.$$

Then, the integration measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ is replaced by the measure $\mathcal{D}\psi_o\mathcal{D}\bar{\psi}_o\mathcal{D}\psi_e\mathcal{D}\bar{\psi}_e$.

To investigate the functional integral (1), we shall perform a successive integration: first, we shall integrate over the high-energy constituents given by (4), and then over the low-energy ones [8,11]. At a second step, it is preferable to pass to new functional variables, which describe an observable "low-energy" physics in a more adequate way. After the substitution of (4) into the action (2) we take into account in S only the terms up to quadratic in $\bar{\psi}_e$, ψ_e. This means an approximation, in which the over-condensate quasi-particles do not couple with each other. In this case, it is possible to integrate out the thermal fluctuations $\bar{\psi}_e$, ψ_e in a closed form and thus to arrive to an effective action functional $S_{\text{eff}}[\psi_o,\bar{\psi}_o]$. It depends only on the quasi-condensate variables $\psi_o, \bar{\psi}_o$:

$$S_{\text{eff}}[\psi_o,\bar{\psi}_o] = \ln \int \mathrm{e}^{\widetilde{S}[\psi_o+\psi_e,\bar{\psi}_o+\bar{\psi}_e]}\mathcal{D}\psi_e\mathcal{D}\bar{\psi}_e\,, \tag{5}$$

where the tilde in $\widetilde{S}$ implies that "self-coupling" of the fields $\bar{\psi}_e$, ψ_e is excluded. With respect of (5), the partition function of the model Z (1) takes an approximate form:

$$Z \approx \int \mathrm{e}^{S_{\text{eff}}[\psi_o,\bar{\psi}_o]}\mathcal{D}\psi_o\mathcal{D}\bar{\psi}_o\,. \tag{6}$$

Let us consider the derivation of the effective action $S_{\text{eff}}[\psi_o,\bar\psi_o]$ (5) in more details. The splitting (4) allows to derive $S_{\text{eff}}[\psi_o,\bar\psi_o]$ in the framework of the field-theoretical approach of loop expansion [15]. We substitute (4) into the initial action $S[\psi,\bar\psi]$ (2) and then go over from S to the action $\widetilde{S}$, which is given by three terms:

$$\widetilde{S} = S_{\text{cond}} + S_{\text{free}} + S_{\text{int}}\,. \tag{7}$$

In (7), S_{cond} is the action functional of the condensate quasi-particles, which corresponds to a tree approximation [15]:

$$\begin{aligned} S_{\text{cond}}[\psi_o,\bar\psi_o] \;\equiv\; \int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \Big\{ & \bar\psi_o(x,\tau)\widehat{K}_+\psi_o(x,\tau) - \\ & -\frac{g}{2}\,\bar\psi_o(x,\tau)\bar\psi_o(x,\tau)\psi_o(x,\tau)\psi_o(x,\tau)\Big\}. \end{aligned} \tag{8}$$

At the chosen approximation the action for the over-condensate excitations S_{free} takes the form:

$$S_{\text{free}}[\psi_e,\bar\psi_e] \equiv \frac{1}{2}\int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \big(\bar\psi_e,\psi_e\big)\widehat{G}^{-1}\begin{pmatrix}\psi_e\\ \bar\psi_e\end{pmatrix}\,, \tag{9}$$

where $\widehat{G}^{-1}$ is the matrix-differential operator,

$$\widehat{G}^{-1} \equiv \widehat{G}_0^{-1} - \widehat{\Sigma}\,. \tag{10}$$

In (10) we defined:

$$\widehat{G}_0^{-1} \equiv \begin{pmatrix}\widehat{K}_+ & 0\\ 0 & \widehat{K}_-\end{pmatrix}, \qquad \widehat{\Sigma} \equiv \widehat{\Sigma}[\psi_o,\bar\psi_o] = g\begin{pmatrix}2\bar\psi_o\psi_o & \psi_o^2\\ (\bar\psi_o)^2 & 2\bar\psi_o\psi_o\end{pmatrix}, \tag{11}$$

where $\widehat{K}_\pm$ are the differential operators, $\widehat{K}_\pm \equiv \pm\dfrac{\partial}{\partial\tau} - \mathcal{H}$, and $\mathcal{H}$ is the single-particle Hamiltonian (3). Eventually, S_{int} describes a coupling of the quasi-condensate to the over-condensate excitations:

$$\begin{aligned} S_{\text{int}}[\psi_o,\bar\psi_o,\psi_e,\bar\psi_e] \;\equiv\; \int_0^\beta \mathrm{d}\tau \int \mathrm{d}x \Big\{ & \bar\psi_e(x,\tau)\big[\widehat{K}_+ - g\bar\psi_o\psi_o\big]\psi_o(x,\tau) + \\ & +\psi_e(x,\tau)\big[\widehat{K}_- - g\bar\psi_o\psi_o\big]\bar\psi_o(x,\tau)\Big\}. \end{aligned} \tag{12}$$

It is appropriate to apply the stationary phase method to the functional integral (6). To this end, let us choose $\bar\psi_o$, ψ_o as the stationarity points of the functional S_{cond} (8), which are defined by the extremum condition $\delta(S_{\text{cond}}[\psi_o,\bar\psi_o]) = 0$. The corresponding equations look like the Gross–Pitaevskii-type equations [1]:

$$\begin{aligned} \left(\frac{\partial}{\partial\tau} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \mu - V(x)\right)\psi_o - g\left(\bar\psi_o\psi_o\right)\psi_o &= 0\,, \\ \left(-\frac{\partial}{\partial\tau} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \mu - V(x)\right)\bar\psi_o - g\left(\bar\psi_o\psi_o\right)\bar\psi_o &= 0\,. \end{aligned} \tag{13}$$

The contribution of the action functional S_{int} (12) drops out from (7), provided $\bar\psi_o$, ψ_o are solutions of equations (13). Therefore the dynamics of ψ_e, $\bar\psi_e$ is described, in the leading

approximation, by the action S_{free} (9). The latter depends on $\bar{\psi}_o$, ψ_o non-trivially through the matrix of the self-energy parts $\widehat{\Sigma}$, which enters into $\widehat{G}^{-1}$ (10).

The *Thomas–Fermi approximation* is essentially used in the present paper in order to determine the stationarity points $\bar{\psi}_o$, ψ_o. This approximation consists in neglect of the kinetic term $\dfrac{\hbar^2}{2m}\dfrac{\partial^2}{\partial x^2}$ in equations (13) [1,2]. The Thomas–Fermi approximation is valid for the systems containing a sufficiently large number of particles in the magneto-optical traps [1,2]. The following condensate τ-independent solution can be obtained:

$$\bar{\psi}_o\psi_o = \rho_{\text{TF}}(x;\mu) \equiv \frac{1}{g}\,\big(\mu - V(x)\big)\Theta\big(\mu - V(x)\big)\,, \tag{14}$$

where Θ is the Heavyside function. Now the integration in (6) with respect to ψ_e, $\bar{\psi}_e$ is Gaussian. This leads to the one loop effective action in terms of the variables ψ_o, $\bar{\psi}_o$:

$$S_{\text{eff}}[\psi_o,\bar{\psi}_o] \equiv S_{\text{cond}}[\psi_o,\bar{\psi}_o] - \tfrac{1}{2}\,\ln\text{Det}\left(\widehat{G}^{-1}\right). \tag{15}$$

Here $\widehat{G}^{-1}$ is the matrix operator (10) and ψ_o, $\bar{\psi}_o$ have a sense of the new variables governed by the action (15).

In order to assign a meaning to the final expression for the effective action (15), it is necessary to regularize the determinant $\text{Det}\left(\widehat{G}^{-1}\right)$. In our case, the operator $\widehat{G}^{-1}$ is already written as 2×2-matrix Dyson equation (10), where the entries of $\widehat{\Sigma}[\psi_o,\bar{\psi}_o]$ (11) play the role of the normal ($\Sigma_{11} = \Sigma_{22}$) and anomalous ($\Sigma_{12}$, Σ_{21}) self-energy parts. The Dyson equation (10) defines the matrix $\widehat{G}$, where the entries have a meaning of the Green functions of the fields $\bar{\psi}_e$, ψ_e. The matrix $\widehat{G}$ arises as a formal inverse of the operator $\widehat{G}^{-1}$.

It is appropriate to represent $\widehat{G}^{-1}$ as follows:

$$\widehat{G}^{-1} = \widehat{G}_0^{-1} - \widehat{\Sigma} \equiv \widehat{\mathcal{G}}^{-1} - \left(\widehat{\Sigma} - 2g\rho_{\text{TF}}(x;\mu)\widehat{I}\right), \tag{16}$$

where $\widehat{I}$ is the unit matrix of the size 2×2, and the matrix $\widehat{\mathcal{G}}^{-1}$ is defined as

$$\widehat{\mathcal{G}}^{-1} \equiv \begin{pmatrix} \widehat{K}_+ - 2g\rho_{\text{TF}}(x;\mu) & 0 \\ 0 & \widehat{K}_- - 2g\rho_{\text{TF}}(x;\mu) \end{pmatrix} \equiv \begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix}. \tag{17}$$

Here $\rho_{\text{TF}}(x;\mu)$ is the solution (14) and equation (16) implies that we simply added and subtracted $2g\rho_{\text{TF}}(x;\mu)$ on the principle diagonal of the matrix operator $\widehat{G}^{-1}$. A formal inverse of $\widehat{\mathcal{G}}^{-1}$ can be found from the following equation, which defines the Green functions $\mathcal{G}_\pm$:

$$\begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix}\begin{pmatrix} \mathcal{G}_+ & 0 \\ 0 & \mathcal{G}_- \end{pmatrix} = \delta(x-x')\delta(\tau-\tau')\widehat{I}\,.$$

Using the relation ln Det = Tr ln, one gets:

$$-\frac{1}{2}\ln\text{Det}\left(\widehat{G}^{-1}\right) = -\frac{1}{2}\,\text{Tr}\,\ln\left(\widehat{I} - \widehat{\mathcal{G}}\big(\widehat{\Sigma} - 2g\rho_{\text{TF}}(x;\mu)\widehat{I}\big)\right) - \frac{1}{2}\ln\text{Det}\begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix}. \tag{18}$$

The first term in right-hand side of (18) is free from divergencies. Let us consider the determinant of the matrix-differential operator in right-hand side of (18). Let us denote the eigenvalues of the operators $\mathcal{K}_\pm$ as $\pm\mathrm{i}\omega_B - \lambda_n$, where ω_B are the bosonic Matsubara

frequencies and λ_n are the energy levels labeled by the multi-index n [5, 6]. Then, we calculate [5,6]:

$$\frac{1}{2\beta}\ln\mathrm{Det}\begin{pmatrix}\mathcal{K}_+ & 0\\ 0 & \mathcal{K}_-\end{pmatrix} = \frac{1}{\beta}\sum_n \ln\left(2\sinh\frac{\beta\lambda_n}{2}\right) \equiv \widetilde{F}_{\mathrm{nc}}(\mu)\,,$$

where $\widetilde{F}_{\mathrm{nc}}$ has a sense of the free energy of an ideal gas of the over-condensate excitations. Then, in the leading order in g, one gets:

$$\begin{aligned}-\frac{1}{2}\ln\mathrm{Det}\left(\widehat{G}^{-1}\right) &\approx -\beta\widetilde{F}_{\mathrm{nc}}(\mu) + \\ &+g\int_0^\beta \mathrm{d}\tau\int \mathrm{d}x\big(\mathcal{G}_+(x,\tau;x,\tau)+\mathcal{G}_-(x,\tau;x,\tau)\big)\left(\bar{\psi}_o\psi_o-\rho_{\mathrm{TF}}(x;\mu)\right)\equiv \\ &\equiv -\beta F_{\mathrm{nc}}(\mu) - 2g\int_0^\beta \mathrm{d}\tau\int \mathrm{d}x\,\rho_{\mathrm{nc}}(x)\bar{\psi}_o\psi_o\,.\end{aligned} \tag{19}$$

Here F_{nc} is the free energy of the non-ideal gas of the over-condensate quasi-particles. The density of the over-condensate quasi-particles is $\rho_{\mathrm{nc}}(x)\equiv -\mathcal{G}_\pm(x,\tau;x,\tau)$, and it depends only on the spatial coordinate x. At very low temperatures and sufficiently far from the boundary of the domain occupied by the condensate, the quantity $\rho_{\mathrm{nc}}(x)$ can approximately be replaced by $\rho_{\mathrm{nc}}(0)$, since $\mathcal{G}_\pm(x,\tau;x,\tau)$ is almost constant over a considerable part of the condensate [16].

It is appropriate to write the one loop effective action obtained in terms of new independent real-valued variables of the functional integration. Namely, in terms of the density $\rho(x,\tau)$ and the phase $\varphi(x,\tau)$ of the field $\psi_o(x,\tau)$:

$$\psi_o(x,\tau)=\sqrt{\rho(x,\tau)}\,\mathrm{e}^{\mathrm{i}\varphi(x,\tau)}\,,\qquad \bar{\psi}_o(x,\tau)=\sqrt{\rho(x,\tau)}\,\mathrm{e}^{-\mathrm{i}\varphi(x,\tau)}\,. \tag{20}$$

In terms of these variables, the effective action takes the form [5,6]:

$$\begin{aligned}S_{\mathrm{eff}}[\rho,\varphi] &= -\beta F_{\mathrm{nc}}(\mu)+\mathrm{i}\int_0^\beta \mathrm{d}\tau\int \mathrm{d}x\left\{\rho\partial_\tau\varphi+\frac{\hbar^2}{2m}\,\partial_x(\rho\partial_x\varphi)\right\}+ \\ &+\int_0^\beta \mathrm{d}\tau\int \mathrm{d}x\left\{\frac{\hbar^2}{2m}\left(\sqrt{\rho}\,\partial_x^2\sqrt{\rho}-\rho(\partial_x\varphi)^2\right)+(\Lambda-V)\rho-\frac{g}{2}\,\rho^2\right\},\end{aligned} \tag{21}$$

where $\Lambda=\mu-2g\rho_{\mathrm{nc}}(0)$ is the renormalized chemical potential. Here and below we denote the partial derivatives of the first order over τ and x as ∂_τ and ∂_x, respectively, whereas the partial derivatives of the second order — as ∂_τ^2 and ∂_x^2. The model in question in the present paper is spatially one-dimensional, and a possible multi-valuedness of the angle variable φ is left aside.

We shall consider S_{eff} (21) as the one loop effective action, where the thermal corrections over the "classical" background are taken into account. The "classical" background corresponds to the solution (14). It should be noticed that our derivation of the effective action can formally be used for two and three dimensions also [4]. Notice that equation (21) remains correct at $V=0$ also.

2.2 The excitation spectrum

Let us determine the spectrum of the low-energy quasi-particles. Now we are applying the stationary phase approximation to the integral (6), where the effective action is given

by (21), while the measure is $\mathcal{D}\rho\mathcal{D}\varphi$. The corresponding stationarity point is given by the extremum condition $\delta\big(S_{\rm eff}[\rho,\varphi]\big)=0$, which is equivalent to the couple of the Gross–Pitaevskii equations:

$$
\begin{aligned}
\mathrm{i}\partial_\tau\varphi+\frac{\hbar^2}{2m}\left(\frac{1}{\sqrt{\rho}}\,\partial_x^2\sqrt{\rho}-(\partial_x\varphi)^2\right)+\Lambda-V(x)-g\rho=0\,,\\
-\mathrm{i}\partial_\tau\rho+\frac{\hbar^2}{m}\,\partial_x\left(\rho\partial_x\varphi\right)=0\,.
\end{aligned}
\tag{22}
$$

We use the Thomas–Fermi approximation and drop out the term $(\partial_x^2\sqrt{\rho})/\sqrt{\rho}$ in the first equation in (22). Solution with $\partial_\tau\rho=0=\partial_\tau\varphi$ appears, provided the velocity field $\mathbf{v}=m^{-1}\partial_x\varphi$ is taken equal to zero in (22). In this case, equations (22) lead to the density of the condensate:

$$
\rho_{\rm TF}(x)\equiv\frac{\Lambda}{g}\,\widetilde{\rho}_{\rm TF}(x)=\frac{\Lambda}{g}\left(1-\frac{x^2}{R_c^2}\right)\Theta\left(1-\frac{x^2}{R_c^2}\right).
\tag{23}
$$

Explicit form of the external potential $V(x)=\frac{1}{2}m\Omega^2x^2$ is taken into account in the expression (23). The form of the solution (23) means that the quasi-condensate occupies the domain $|x|\le R_c$ at zero temperature. The length R_c defines the boundary of this domain, $R_c^2\equiv\dfrac{2\Lambda}{m\Omega^2}$ (in three dimensional space, this would correspond to a spherical distribution of the condensate). In the homogeneous case given by the limit $1/R_c\to0$, the Thomas–Fermi solution $\rho_{\rm TF}(x)$ is transformed into the density $\rho_{\rm TF}(0)=\Lambda/g$, which coincides with the density of the homogeneous Bose gas [8].

According to the initial splitting (4), we suppose that thermal fluctuations in vicinity of the stationarity point (23) are small, and therefore an analogous splitting can be written for the condensate density also:

$$
\rho_0(x,\tau)=\rho_{\rm TF}(x)+\pi_0(x,\tau)\,,
\tag{24}
$$

where $\rho_0(x,\tau)$ is a specific solution of (22). We linearize equations (22) in a vicinity of the equilibrium solution $\rho_0=\rho_{\rm TF}(x)$, $\varphi=\text{const}$. Eliminating the phase φ and dropping out the terms proportional to $\hbar^4$, we go over from (22) to the *Stringari thermal equation* [17]:

$$
\frac{1}{\hbar^2v^2}\,\partial_\tau^2\pi_0+\partial_x\bigg(\Big(1-\frac{x^2}{R_c^2}\Big)\partial_x\pi_0\bigg)=0\,,
\tag{25}
$$

where the parameter v has a meaning of the sound velocity in the center of the trap:

$$
v^2\equiv\frac{\rho_{\rm TF}(0)g}{m}=\frac{\Lambda}{m}\,.
\tag{26}
$$

The substitution $\pi_0=\mathrm{e}^{\mathrm{i}\omega\tau}u(x)$ transforms (25) into the Legendre equation:

$$
-\frac{\omega^2}{\hbar^2v^2}\,u(x)+\frac{\mathrm{d}}{\mathrm{d}x}\bigg(\Big(1-\frac{x^2}{R_c^2}\Big)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\bigg)=0\,.
\tag{27}
$$

Since the Thomas–Fermi solution (23) is non-zero only at $|x|\le R_c$, we shall consider (27) at $x\in[-R_c,R_c]\subset\mathbb{R}$, as well. After an analytical continuation $\omega\to\mathrm{i}E$, equation (27) possesses the polynomial solutions, which are given by the Legendre polynomials $P_n(x/R_c)$, if and only if

$$
\left(\frac{R_c}{\hbar v}\right)^2E^2\equiv\frac{2}{\hbar^2\Omega^2}\,E^2=n(n+1)\,,\qquad n\ge0\,.
\tag{28}
$$

In other words, equation (27) leads to the spectrum of the low lying excitations: $E_n = \hbar\Omega\sqrt{\frac{n(n+1)}{2}}$, $n \geq 0$, [18]. Notice that the corresponding equation for the homogeneous Bose gas is obtained after a formal limit $1/R_c \to 0$ in (27) at finite x. Provided the latter is still considered for the segment $[-R_c, R_c] \ni x$ with a periodic boundary condition for x, we arrive at the discrete spectrum of the following form: $E_k = \hbar v k$, where k is the wave number, $k = (\pi/R_c)n$, $n \in \mathbb{Z}$.

3 The two-point correlation functions

Let us go over to our main task — to the calculation of the two-point thermal correlation function $\Gamma(x_1, \tau_1; x_2, \tau_2)$ of the spatially non-homogeneous Bose gas. We define it as the ratio of two functional integrals:

$$\Gamma(x_1, \tau_1; x_2, \tau_2) = \frac{\int \bar{\psi}(x_1, \tau_1)\psi(x_2, \tau_2)\mathrm{e}^{S[\psi,\bar{\psi}]}\mathcal{D}\psi\mathcal{D}\bar{\psi}}{\int \mathrm{e}^{S[\psi,\bar{\psi}]}\mathcal{D}\psi\mathcal{D}\bar{\psi}}\,, \tag{29}$$

where the action $S[\psi, \bar{\psi}]$ is given by (2).

We are interested in the behaviour of the correlators at the distances considerably smaller in comparison with the size of the domain occupied by the condensate. The main contribution to the behaviour of the correlation functions is due to the low lying excitations at sufficiently low temperatures [8,11]. To calculate $\Gamma(x_1, \tau_1; x_2, \tau_2)$, (29), we use the method of successive functional integration first over the high-energy excitations $\bar{\psi}_e$, ψ_e, and then over the low-energy excitations $\bar{\psi}_o$, ψ_o (see (4)). In the leading approximation, the correlator we are interested in looks, in terms of the density–phase variables, as follows [5,6]:

$$\begin{aligned}\Gamma(x_1, \tau_1; x_2, \tau_2) \;\simeq\; & \int \exp\Big(S_{\text{eff}}[\rho, \varphi] - \mathrm{i}\varphi(x_1, \tau_1) + \mathrm{i}\varphi(x_2, \tau_2) + \\ & \qquad + \tfrac{1}{2}\ln\rho(x_1, \tau_1) + \tfrac{1}{2}\ln\rho(x_2, \tau_2)\Big)\mathcal{D}\rho\mathcal{D}\varphi \times \\ & \times \Big(\int \exp\big(S_{\text{eff}}[\rho, \varphi]\big)\mathcal{D}\rho\mathcal{D}\varphi\Big)^{-1},\end{aligned} \tag{30}$$

where the integrand in the nominator is arranged in the form of a single exponential. Here $S_{\text{eff}}[\rho, \varphi]$ is the effective action (21).

Since the fluctuations of the density are suppressed at sufficiently low temperatures [16], one can replace $\ln\rho(x_1, \tau_1)$, $\ln\rho(x_2, \tau_2)$ in (30) by $\ln\rho_{\text{TF}}(x_1)$, $\ln\rho_{\text{TF}}(x_2)$, where ρ_{TF} is defined by (23). In accordance with the variational principle suggested in [14], we estimate the functional integrals in (30) by the stationary phase method. Each of the integrals is characterized by its own stationarity point given by variation of the corresponding exponent. For the correlation function $\Gamma(x_1, \tau_1; x_2, \tau_2)$, we obtain the following leading estimation:

$$\begin{aligned}\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq & \sqrt{\rho_{\text{TF}}(x_1)\rho_{\text{TF}}(x_2)} \times \\ & \times \exp\big(-S_{\text{eff}}[\rho_0, \varphi_0] + S_{\text{eff}}[\rho_1, \varphi_1] - \mathrm{i}\varphi_1(x_1, \tau_1) + \mathrm{i}\varphi_1(x_2, \tau_2)\big)\,,\end{aligned} \tag{31}$$

where the variables ρ_0, φ_0 are defined by the extremum condition $\delta\big(S_{\text{eff}}[\rho, \varphi]\big) = 0$, and therefore they just satisfy the Gross–Pitaevskii equations (22). The fields ρ_1, φ_1 are defined by the extremum condition:

$$\delta\Big(S_{\text{eff}}[\rho, \varphi] - \mathrm{i}\varphi(x_1, \tau_1) + \mathrm{i}\varphi(x_2, \tau_2)\Big) = 0\,. \tag{32}$$

The variational equation (32) leads to another couple of equations of the Gross–Pitaevskii type. One of these equations turns out to be a non-homogeneous equation with the δ-like source, while another one is a homogeneous equation. In fact, the homogeneous equation appears due to a requirement of vanishing of the coefficient at the variation $\delta\rho(x,\tau)$, while the non-homogeneous equation is defined by vanishing of the coefficient at the variation $\delta\varphi(x,\tau)$.

It can consistently be assumed that the solution $\rho_1(x,\tau)$ can be represented as a sum of $\rho_{\rm TF}(x)$ and of a weakly fluctuating part, provided the boundary R_c is far from beginning of coordinates: $\rho_1(x,\tau) = \rho_{\rm TF}(x) + \pi_1(x,\tau)$. Therefore, the terms $\sqrt{\pi_1}\,\partial_x^2\sqrt{\pi_1}$ and $\partial_x\pi_1\partial_x\varphi_1$ are small and can be omitted. Taking into account a linearization near the Thomas–Fermi solution, one can finally arrive at a couple of the following equations:

$$\begin{aligned} \mathrm{i}\partial_\tau\varphi_1 - g\pi_1 - \frac{\hbar^2}{2m}\left(\partial_x\varphi_1\right)^2 &= 0\,, \\ -\mathrm{i}\partial_\tau\pi_1 + \frac{\hbar^2}{m}\,\partial_x\left(\rho_{\rm TF}\partial_x\varphi_1\right) &= \mathrm{i}\delta(x-x_1)\delta(\tau-\tau_1) - \mathrm{i}\delta(x-x_2)\delta(\tau-\tau_2)\,. \end{aligned} \tag{33}$$

Equations (33) lead [5,6] to the following equation for the variable φ_1:

$$\frac{1}{\hbar^2 v^2}\,\partial_\tau^2\varphi_1 + \partial_x\left(\widetilde{\rho}_{\rm TF}(x)\partial_x\varphi_1\right) = \mathrm{i}\,\frac{mg}{\hbar^2\Lambda}\Big\{\delta(x-x_1)\delta(\tau-\tau_1) - \delta(x-x_2)\delta(\tau-\tau_2)\Big\}, \tag{34}$$

where v means the sound velocity in the center of the trap (26), and $\widetilde{\rho}_{\rm TF}$ is defined by (23). Now, with the help of (33) one can calculate the terms contributing into the exponent in (31), [5,6]:

$$-S_{\rm eff}[\rho_0,\varphi_0] + S_{\rm eff}[\rho_1,\varphi_1] \simeq \frac{\mathrm{i}}{2}\,\big(\varphi_1(x_1,\tau_1) - \varphi_1(x_2,\tau_2)\big)\,. \tag{35}$$

Substituting (35) into (31), one obtains the following approximate formula for the correlator:

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \sqrt{\rho_{\rm TF}(x_1)\rho_{\rm TF}(x_2)}\,\exp\left(-\frac{\mathrm{i}}{2}\,\big(\varphi_1(x_1,\tau_1) - \varphi_1(x_2,\tau_2)\big)\right). \tag{36}$$

It is natural to represent solutions of equation (34) in terms of the solution $G(x,\tau;x',\tau')$ of the equation

$$\frac{1}{\hbar^2 v^2}\,\partial_\tau^2 G(x,\tau;x',\tau') + \partial_x\left(\Big(1-\frac{x^2}{R_c^2}\Big)\partial_x G(x,\tau;x',\tau')\right) = \frac{g}{\hbar^2 v^2}\,\delta(x-x')\delta(\tau-\tau')\,. \tag{37}$$

Bearing in mind the homogeneous equation (25), we shall call (37) as *non-homogeneous Stringari equation*. As it is clear after [19], the Green function $G(x_1,\tau_1;x_2,\tau_2)$ has a meaning of the correlation function of the phases:

$$G(x_1,\tau_1;x_2,\tau_2) = -\big\langle\varphi(x_1,\tau_1)\varphi(x_2,\tau_2)\big\rangle, \tag{38}$$

where the angle brackets in right-hand side should be understood as an averaging with respect to the weighted measure $\exp\big(S_{\rm eff}[\rho,\varphi]\big)\,\mathcal{D}\rho\mathcal{D}\varphi$. Using (38), it is possible to represent, eventually, the correlation function as follows [5,6]:

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}\,\exp\left(-\tfrac{1}{2}\,\big(G(x_1,\tau_1;x_2,\tau_2) + G(x_2,\tau_2;x_1,\tau_1)\big)\right), \tag{39}$$

where $\widetilde{\rho}(x_1)$, $\widetilde{\rho}(x_2)$ are the renormalized densities. The solution $G(x_1,\tau_1;x_2,\tau_2)$ of equation (37) is defined up to a purely imaginary additive constant, which has a meaning of a global phase.

The governing equations reported in [4] for the spatial dimensionalities $d = 3,\, 2,\, 1$ are in a direct agreement with (33), provided the τ-dependence is neglected in (33). The correlation functions of the phases are obtained in [4] without an influence of $\partial_x \rho_{\mathrm{TF}}\, \partial_x \varphi_1$ as follows (the notation $f(\mathbf{x}, \mathbf{x}')$ is used for them in [4], but $G(\mathbf{x}, \mathbf{x}')$ is used below to keep contact with (38)):

$$G(\mathbf{x}, \mathbf{x}') = -\frac{\Lambda}{4\pi\beta\hbar^2 v^2 \rho_{\mathrm{TF}}(S)} \frac{1}{\mid \mathbf{x} - \mathbf{x}' \mid} \,, \qquad (d = 3) \,, \tag{40a}$$

$$G(\mathbf{x}, \mathbf{x}') = \frac{\Lambda}{2\pi\beta\hbar^2 v^2 \rho_{\mathrm{TF}}(S)} \ln \frac{\mid \mathbf{x} - \mathbf{x}' \mid}{\lambda_T} \,, \qquad (d = 2) \,, \tag{40b}$$

$$G(\mathbf{x}, \mathbf{x}') = \frac{\Lambda}{2\beta\hbar^2 v^2 \rho_{\mathrm{TF}}(S)} \mid \mathbf{x} - \mathbf{x}' \mid , \qquad (d = 1) \,. \tag{40c}$$

where $\mathbf{x}$, $\mathbf{x}'$ label spatial arguments at $d = 3,\, 2,\, 1$ and $S \equiv \frac{1}{2}\,(\mathbf{x} + \mathbf{x}')$. In (40b), the thermal length $\lambda_T = \hbar\beta v$ is introduced, where $v = \sqrt{\Lambda/m}$ is the sound velocity given by (26). It is already clear that the correlation functions can no longer depend on $\mid \mathbf{x} - \mathbf{x}' \mid$ alone: they depend also on the center of mass coordinate S, consistent with the breakdown of translational invariance induced by the trap.

Let us remind first the correlation functions in $d = 3,\, 2$. In this case, the points $\mathbf{x} = \mathbf{x}_1$ and $\mathbf{x} = \mathbf{x}_2$ in $\varphi_1 \equiv \varphi_1(\mathbf{x}; \mathbf{x}_1, \mathbf{x}_2)$ are singular and introduce a divergence problem [8]. This difficulty can be avoided by considering a first-order *coherence function* $\Gamma^{(1)}(\mathbf{x}_1, \mathbf{x}_2)$ which is defined in [4] as

$$\Gamma^{(1)}(\mathbf{x}_1, \mathbf{x}_2) \simeq \frac{\Gamma(\mathbf{x}_1, \mathbf{x}_2)}{\langle \psi_o(\mathbf{x}_1) \rangle \, \langle \bar{\psi}_o(\mathbf{x}_2) \rangle} \,, \tag{41}$$

where $\Gamma(\mathbf{x}_1, \mathbf{x}_2)$ is given by (36). Then, $\Gamma^{(1)}(\mathbf{x}_1, \mathbf{x}_2)$ is both finite and well defined because identically the same singularities appear [8] in a direct calculation of $\langle \psi_o(\mathbf{x}_1) \rangle$ and $\langle \bar{\psi}_o(\mathbf{x}_2) \rangle$. We find that

$$\Gamma^{(1)}(\mathbf{x}_1, \mathbf{x}_2) \simeq \exp\left(\frac{\Lambda}{4\pi\beta\hbar^2 v^2 \rho_{\mathrm{TF}}(S)} \frac{1}{\mid \mathbf{x}_1 - \mathbf{x}_2 \mid} \right) . \tag{42}$$

Evidently for $d = 3$,

$$\Gamma^{(1)}(\mathbf{x}_1, \mathbf{x}_2) \;\longrightarrow\; 1 + \frac{\Lambda}{4\pi\beta\hbar^2 v^2 \rho_{\mathrm{TF}}(S)} \frac{1}{\mid \mathbf{x}_1 - \mathbf{x}_2 \mid}$$

for $\mid \mathbf{x}_1 - \mathbf{x}_2 \mid \gg \Lambda/(4\pi\beta\hbar^2 v^2 \rho_{\mathrm{TF}}(S))$, thus indicating long-range order and long-range coherence. The *correlation length* given by $\Lambda/4\pi\beta\hbar^2 v^2 \rho_{\mathrm{TF}}(S))$ is therefore a slowly varying function of $\mathbf{x}_1$ and $\mathbf{x}_2$. Notice that we have assumed $\mathbf{x}_1$ and $\mathbf{x}_2$ are not close to the boundaries.

In the case $d = 2$, the correlations decay by a power law for arbitrary small temperatures,

$$\Gamma^{(1)}(\mathbf{x}_1, \mathbf{x}_2) \simeq \left(\frac{\lambda_T}{\mid \mathbf{x}_1 - \mathbf{x}_2 \mid} \right)^{\Lambda/(2\pi\beta\hbar^2 v^2 \rho_{\mathrm{TF}}(S))} . \tag{43}$$

The exponent of this power-law is proportional to T so that, at very low temperatures, correlations may thus prevail over almost macroscopic distances. At $T = 0$ exactly, $\Gamma^{(1)}$ will have a non-zero off-set due to the presence of a true condensate.

For $d=1$, one can obtain for G instead of (40c) an exact expression as follows (i.e., the term $\partial_x\rho_{\rm TF}\,\partial_x\varphi_1$ in the governing equations is now accounted for):

$$G(\mathbf{x},\mathbf{x}')\equiv G(x,x')=\frac{gR_c}{\beta(2\hbar v)^2}\,\ln\left[\frac{\left(1+|x-x'|/(2R_c)\right)^2-(x+x')^2/(4R_c^2)}{\left(1-|x-x'|/(2R_c)\right)^2-(x+x')^2/(4R_c^2)}\right]. \tag{44}$$

Then, we obtain $\Gamma(\mathbf{x}_1,\mathbf{x}_2)\equiv\Gamma(x_1,x_2)$ in the following form:

$$\Gamma(x_1,x_2)\simeq\sqrt{\rho_{\rm TF}(x_1)\rho_{\rm TF}(x_2)}\left[\frac{1+|x_1-x_2|/R_c-x_1x_2/R_c^2}{1-|x_1-x_2|/R_c-x_1x_2/R_c^2}\right]^{-gR_c/(4\beta\hbar^2v^2)}. \tag{45}$$

In the limit $|\,\mathbf{x}_1-\mathbf{x}_2\,|\ll\big\{R_c,S\big\}$, we obtain $\Gamma(x_1,x_2)$ in the form:

$$\Gamma(x_1,x_2)\simeq\sqrt{\rho_{\rm TF}(x_1)\rho_{\rm TF}(x_2)}\;\exp\left(-\frac{\Lambda}{2\beta\hbar^2v^2\rho_{\rm TF}(S)}\,|x_1-x_2|\right). \tag{46}$$

The correlation length depends now on x_1 and x_2. Without the trap, the correlation length reduces to $2\beta\hbar^2\rho_{\rm TF}(0)/m$, where $\rho_{\rm TF}(0)$ is the density of the ground state of a homogeneous system at zero temperature, in complete agreement with the exact solution [3].

It is obvious that the correlation functions (43) and (46) vanish for large separation of the arguments at arbitrary small temperatures $T>0$, and that there is no long-range order in $d=2$ or $d=1$. The correlation functions (42), (43), (46) coincide with those obtained under translational invariance without the trap to the extent that there is now an additional factor $\rho_{\rm TF}(S)$ in the exponents. True long-range order arises only in $d=3$.

4 The asymptotics of the correlation functions

Therefore, the problem concerning the study of the asymptotical behaviour of the two-point thermal correlation function $\Gamma(x_1,\tau_1;x_2,\tau_2)$ given by the representation (39), is reduced to solution of the non-homogeneous Stringari equation (37). The corresponding answer (or its asymptotics) should be subsequently substituted into (39). In the present section, we shall obtain explicitly solutions of (37), and we shall consider the corresponding asymptotics of $\Gamma(x_1,\tau_1;x_2,\tau_2)$. Let us begin with the limiting case of a homogeneous Bose gas.

4.1 The homogeneous Bose gas

The homogeneous case is given by $V(x)\equiv 0$, and the related equation appears from (37) at $1/R_c\to 0$:

$$\frac{1}{\hbar^2v^2}\,\partial_\tau^2G(x,\tau;x',\tau')+\partial_x^2G(x,\tau;x',\tau')=\frac{g}{\hbar^2v^2}\,\delta(x-x')\delta(\tau-\tau'). \tag{47}$$

We consider (47) for the domain $[-R_c,R_c]\times[0,\beta]\ni(x,\tau)$ with the periodic boundary conditions for each variable. The δ-functions in right-hand side of (47) are treated as the periodic δ-functions. This allows us to represent the solution of this equation as the formal double Fourier series:

$$G(x,\tau;x',\tau')=\left(\frac{-g}{2\beta R_c}\right)\sum_{\omega,k}\frac{\mathrm{e}^{\mathrm{i}\omega(\tau-\tau')+\mathrm{i}k(x-x')}}{\omega^2+E_k^2}, \tag{48}$$

where $\omega = (2\pi/\beta)l$, $l \in \mathbb{Z}$. The notation for the energy $E_k = \hbar v k$, where $k = (\pi/R_c)n$, $n \in \mathbb{Z}$, is used in (48). Besides, the representation (48) requires a regularization, which consists in neglect of the term given by $\omega = k = 0$.

Let us deduce from (48) two important asymptotical representations for the Green function. Then, in the limit of zero temperature and of infinite size of the domain occupied by the Bose gas, one can go over to the asymptotics of $\Gamma(x_1, \tau_1; x_2, \tau_2)$. When a strong inequality $\beta^{-1} \equiv k_B T \gg \hbar v/R_c$ is valid, we obtain:

$$\begin{aligned} &G(x, \tau; x', \tau') \simeq \\ &\simeq \frac{g}{2\pi\hbar v} \ln\left\{2\left|\sinh\frac{\pi}{\hbar\beta v}\big(|x - x'| + \mathrm{i}\hbar v(\tau - \tau')\big)\right|\right\} - \frac{g}{4\beta R_c}\frac{|x - x'|^2}{\hbar^2 v^2} + \mathcal{C}\,, \end{aligned} \tag{49}$$

where $|x - x'| \leq 2R_c$, $|\tau - \tau'| \leq \beta$, and $\mathcal{C}$ is some constant, which is not written explicitly. When an opposite inequality $\beta^{-1} \equiv k_B T \ll \hbar v/R_c$ is valid, we obtain:

$$\begin{aligned} &G(x, \tau; x', \tau') \simeq \\ &\simeq \frac{g}{2\pi\hbar v} \ln\left\{2\left|\sinh\frac{\mathrm{i}\pi}{2R_c}\big(|x - x'| + \mathrm{i}\hbar v(\tau - \tau')\big)\right|\right\} - \frac{g}{4\beta R_c}|\tau - \tau'|^2 + \mathcal{C}'\,, \end{aligned} \tag{50}$$

where $|x - x'| \leq 2R_c$, $|\tau - \tau'| \leq \beta$, and $\mathcal{C}'$ is another constant.

Let us substitute the estimate (49) into the representation (39) and take simultaneously the limit $\beta\hbar v/R_c \to 0$ (the size is growing faster than inverse temperature). Then, we obtain the following expression for the correlator in question:

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}\,\left|\sinh\frac{\pi}{\hbar\beta v}\big(|x_1 - x_2| + \mathrm{i}\hbar v(\tau_1 - \tau_2)\big)\right|^{-g/2\pi\hbar v}. \tag{51}$$

Further, applying the relation (50) and taking the limit $R_c/(\beta\hbar v) \to 0$ (the inverse temperature grows faster than the size), we obtain for $\Gamma(x_1, \tau_1; x_2, \tau_2)$:

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}\,\left|\sinh\frac{\mathrm{i}\pi}{2R_c}\big(|x_1 - x_2| + \mathrm{i}\hbar v(\tau_1 - \tau_2)\big)\right|^{-g/2\pi\hbar v}. \tag{52}$$

It follows from (51) and (52), that in the limit of zero temperature, $(\hbar\beta v)^{-1} \to 0$, and of infinite size, $1/R_c \to 0$, the two-point correlation function behaves like

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \frac{\sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}}{\big||x_1 - x_2| + \mathrm{i}\hbar v(\tau_1 - \tau_2)\big|^{1/\theta}}\,. \tag{53}$$

The latter formula is valid in the limit $\beta\hbar v/R_c \to 0$, as well as in the limit $R_c/(\beta\hbar v) \to 0$. In (53) θ denotes the critical exponent: $\theta \equiv 2\pi\hbar v/g$, and the arguments x_1 and x_2, τ_1 and τ_2 are assumed to be sufficiently close each to other. Using the notations $v = \sqrt{\Lambda/m}$ for the sound velocity and $\rho = \Lambda/g$ for the density of the homogeneous Bose gas, we obtain for the critical exponent the following universal expression:

$$\theta = \frac{2\pi\hbar\rho}{mv}\,. \tag{54}$$

4.2 The trapped Bose gas. High temperature case: $k_B T \gg \hbar v / R_c$

Let us turn to non-homogeneous Bose gas described by equations (1)–(3). Now, we should consider the non-homogeneous Stringari equation (37) for the arguments $(x, \tau) \in [-R_c, R_c] \times [0, \beta]$ with the periodic boundary condition only respectively to τ (contrary to equation (47), $\delta(x - x')$ is a usual Dirac's δ-function supported at the point $x' \in \mathbb{R}$). The Green function satisfying (37) can be written as a formal Fourier series:

$$G(x, \tau; x', \tau') = \frac{1}{\beta} \sum_{\omega} \mathrm{e}^{\mathrm{i}\omega(\tau - \tau')} G_\omega(x, x') \,, \tag{55}$$

where $\omega = (2\pi/\beta) l$, $l \in \mathbb{Z}$. The spectral density $G_\omega(x, x')$ in (55) is then governed by the equation

$$-\frac{\omega^2}{\hbar^2 v^2} G_\omega(x, x') + \frac{\mathrm{d}}{\mathrm{d}x} \left(\left(1 - \frac{x^2}{R_c^2} \right) \frac{\mathrm{d}}{\mathrm{d}x} G_\omega(x, x') \right) = \frac{g}{\hbar^2 v^2} \, \delta(x - x') \,. \tag{56}$$

Solution of equation (56) can be obtained in terms of the Legendre functions of the first and second kind, $P_\nu(x/R_c)$ and $Q_\nu(x/R_c)$, [20], which are linearly independent solutions of the homogeneous Legendre equation (27). As a result we get [5, 6]:

$$G_\omega(x, x') = \Re G_\omega(x, x') + \mathrm{i} \Im G_\omega(x, x') \,, \tag{57}$$

where

$$\begin{aligned} \Re G_\omega(x, x') &= \frac{g R_c}{2\hbar^2 v^2} \, \epsilon(x - x') \Big\{ Q_\nu\Big(\frac{x}{R_c}\Big) P_\nu\Big(\frac{x'}{R_c}\Big) - Q_\nu\Big(\frac{x'}{R_c}\Big) P_\nu\Big(\frac{x}{R_c}\Big) \Big\}, \\ \Im G_\omega(x, x') &= -\frac{g R_c}{2\hbar^2 v^2} \Big\{ \frac{2}{\pi} \, Q_\nu\Big(\frac{x}{R_c}\Big) Q_\nu\Big(\frac{x'}{R_c}\Big) + \frac{\pi}{2} \, P_\nu\Big(\frac{x'}{R_c}\Big) P_\nu\Big(\frac{x}{R_c}\Big) \Big\}, \end{aligned} \tag{58}$$

ν looks as follows:

$$\nu = -\frac{1}{2} + \sqrt{\frac{1}{4} - \Big(\frac{R_c}{\hbar v}\Big)^2 \omega^2} \,,$$

and $\epsilon(x - x')$ is the sign function $\epsilon(x) \equiv \mathrm{sign}(x)$. Validity of the solution (57), (58) can be verified by direct substitution into (56), where an expression for the Wronskian of two linearly independent solutions P_ν and Q_ν [21] should be used.

The Green function (57) can be represented in the form, which allows to study the corresponding asymptotical behaviour. When the coordinates x_1, x_2 are chosen to be far from the boundary of the trap, $x_1, x_2 \ll R_c$, but at the same time the inequalities $|x_1 - x_2| \ll \frac{1}{2}(x_1 + x_2)$ and $|x_1 - x_2| \ll R_c$ are valid, the corresponding limit should be called as *quasi-homogeneous*. In the case of strong inequality $\beta^{-1} = k_B T \gg \hbar v / R_c$, we approximately obtain for non-zero frequencies: $|\omega| \gg \hbar v / (2R_c)$. Using the standard asymptotics of the Legendre functions [20, 22], we determine the behaviour of $G_\omega(x, x')$ in the quasi-homogeneous limit at large $|\omega|$ as follows:

$$G_\omega(x, x') \simeq -\frac{\Lambda}{2\hbar v \rho_{\mathrm{TF}}(S)} \, \frac{\exp\big(-(\hbar v)^{-1} |\omega| |x - x'|\big)}{|\omega|} \,. \tag{59}$$

Here S means a half-sum of the spatial arguments of the correlator, $S \equiv \frac{1}{2}(x_1 + x_2)$, and v is given by (26).

Further, we find that the term $\beta^{-1}G_0(x,x')$ in (55) is just given (in the quasi-homogeneous limit) by $G(\mathbf{x},\mathbf{x}')$ (40c). Using (40c) and (59) for evaluation of the series (55), one obtains the answer:

$$G(x,\tau;x',\tau') \simeq \frac{\Lambda}{2\pi\hbar v \rho_{\mathrm{TF}}(S)} \ln\left\{2\left|\sinh\frac{\pi}{\hbar\beta v}\bigl(|x-x'|+\mathrm{i}\hbar v(\tau-\tau')\bigr)\right|\right\}. \tag{60}$$

Therefore, the Green function (39) takes the following form at $\beta^{-1} \gg \hbar v/R_c$:

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \frac{\sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}}{\left|\sinh\frac{\pi}{\hbar\beta v}\bigl(|x_1-x_2|+\mathrm{i}\hbar v(\tau_1-\tau_2)\bigr)\right|^{1/\theta(S)}}, \tag{61}$$

where the critical exponent $\theta(S)$ depends now only on the half-sum of the coordinates S:

$$\theta(S) = \frac{2\pi\hbar\rho_{\mathrm{TF}}(S)}{mv}. \tag{62}$$

The result (61), which is valid for the spatially non-homogeneous case, is in a correspondence with the estimation (51) obtained above for the homogeneous Bose gas. Therefore, the expression (61) is also concerned with validity of the condition that the size of the domain occupied by the Bose condensate grows faster than inverse temperature, i.e., with the condition $\hbar\beta v/R_c \to 0$.

The relation (61) can be simplified for two important limiting cases. Provided the condition

$$1 \ll \frac{|x_1-x_2|}{\hbar\beta v} \ll \frac{R_c}{\hbar\beta v} \tag{63}$$

is fulfilled in the quasi-homogeneous case, we obtain from (61) that the correlator decays exponentially:

$$\begin{aligned}\Gamma(x_1,\tau_1;x_2,\tau_2) &\simeq \sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}\,\exp\left(-\frac{1}{\xi(S)}\bigl||x_1-x_2|+\mathrm{i}\hbar v(\tau_1-\tau_2)\bigr|\right),\\ \xi^{-1}(S) &= \frac{\Lambda}{2\beta\hbar^2 v^2\rho_{\mathrm{TF}}(S)}.\end{aligned} \tag{64}$$

The correlation length $\xi(S)$ is defined by the relation (64), which depends now on the half-sum of the coordinates:

$$\xi(S) \equiv \frac{\hbar\beta v}{\pi}\,\theta(S) = \frac{2\hbar^2\beta\rho_{\mathrm{TF}}(S)}{m}. \tag{65}$$

In an opposite limit,

$$\frac{|x_1-x_2|}{\hbar\beta v},\ \frac{|\tau_1-\tau_2|}{\beta} \ll 1 \ll \frac{R_c}{\hbar\beta v}, \tag{66}$$

the asymptotics of $\Gamma(x_1,\tau_1;x_2,\tau_2)$ takes the following form:

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \frac{\sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}}{\bigl||x_1-x_2|+\mathrm{i}\hbar v(\tau_1-\tau_2)\bigr|^{1/\theta(S)}}. \tag{67}$$

The obtained asymptotics (67) is analogous to the estimation (53), which characterizes the spatially homogeneous Bose gas. But the critical exponent $\theta(S)$ in (67) differs from θ (54), since the latter does not depend on the spatial coordinates.

4.3 The trapped Bose gas. Low temperature case: $k_BT \ll \hbar v/R_c$

Let us go over to another case, which also admits investigation of the asymptotical behaviour of the two-point correlator $\Gamma(x_1,\tau_1;x_2,\tau_2)$. We begin with the non-homogeneous Stringari equation (37), which can be rewritten in the following form:

$$\begin{aligned} \partial^2_\tau G(x,\tau;x',\tau') &+ \frac{1}{\alpha^2}\,\partial_{(x/R_c)}\Bigl(\Bigl(1-\frac{x^2}{R_c^2}\Bigr)\partial_{(x/R_c)}G(x,\tau;x',\tau')\Bigr) = \\ &= \frac{g}{R_c}\,\delta\Bigl(\tfrac{x-x'}{R_c}\Bigr)\delta(\tau-\tau')\,, \end{aligned} \tag{68}$$

where the notation $\alpha \equiv R_c/(\hbar v)$ is introduced. The asymptotical behaviour can be investigated in two cases: $\beta/\alpha \ll 1$ (the previous subsection) and $\beta/\alpha \gg 1$ (see below). The functions

$$\sqrt{n+\frac{1}{2}}\,P_n\Bigl(\frac{x}{R_c}\Bigr)\,, \quad n\geq 0\,,$$

where $P_n(x/R_c)$ are the Legendre polynomials, constitute a complete orthonormal system in the space $L_2\,[-R_c,R_c]$. This fact allows to obtain the following representation for the Green function $G(x,\tau;x',\tau')$ in the form of the generalized double Fourier series:

$$G(x,\tau;x',\tau') = \Bigl(\frac{-g}{\beta R_c}\Bigr)\sum_{\omega}\sum_{n=0}^{\infty}\frac{n+\frac{1}{2}}{\omega^2+E_n^2}\,P_n\Bigl(\frac{x}{R_c}\Bigr)P_n\Bigl(\frac{x'}{R_c}\Bigr)\mathrm{e}^{\mathrm{i}\omega(\tau-\tau')}. \tag{69}$$

Summation $\sum\limits_{\omega}$ in (69), as well as in (48) and (55), goes over the Bose frequencies, and the following notation for the energy levels (28) is adopted:

$$E_n = \hbar\Omega\sqrt{\frac{n(n+1)}{2}} = \frac{\sqrt{n(n+1)}}{\alpha}\,. \tag{70}$$

After summation over the frequencies and after regularization consisting in neglect of the term corresponding to zero values of ω and n, $G(x,\tau;x',\tau')$ (69) takes the form:

$$\begin{aligned} G(x,\tau;x',\tau') = \Bigl(\frac{-g}{\beta R_c}\Bigr)\Bigl[\Bigl(\frac{\beta}{2\pi}\Bigr)^2\sum_{l=1}^{\infty}\frac{\cos\bigl(\frac{2\pi\Delta\tau}{\beta}\,l\bigr)}{l^2} + \\ +\frac{\beta}{2}\sum_{n=1}^{\infty}\frac{n+\frac{1}{2}}{E_n}\,P_n\Bigl(\frac{x}{R_c}\Bigr)P_n\Bigl(\frac{x'}{R_c}\Bigr)\Bigl(\coth\Bigl(\frac{1}{2}\,\beta E_n\Bigr)\cosh(E_n\Delta\tau)-\sinh(E_n\Delta\tau)\Bigr)\Bigr], \end{aligned} \tag{71}$$

where $\Delta\tau \equiv |\tau-\tau'|$. The representation (71) can be studied in both cases: $\beta/\alpha \ll 1$ and $\beta/\alpha \gg 1$. For instance, using (71) at coinciding arguments $\tau=\tau'$ to obtain $\Gamma(x_1,\tau;x_2,\tau)$ in the limit $\beta/\alpha \ll 1$, we just obtain [6] the representations (45) or (under the quasi-homogeneity condition) (46).

Let us turn to the case $\beta/\alpha \gg 1$, where $\beta E_n \gg 1$, $\forall n\geq 1$. In other words, let us suppose that $k_BT \ll E_n$ and, so, $k_BT \ll \hbar\Omega$. Then, one obtains from (71):

$$\begin{aligned} G(x,\tau;x',\tau') = \frac{-g\beta}{4R_c}\Bigl[\Bigl(\frac{1}{2}-\frac{\Delta\tau}{\beta}\Bigr)^2-\frac{1}{12}\Bigr] - \\ -\frac{g}{2\hbar v}\sum_{n=1}^{\infty}\frac{n+\frac{1}{2}}{\sqrt{n(n+1)}}\,P_n\Bigl(\frac{x}{R_c}\Bigr)P_n\Bigl(\frac{x'}{R_c}\Bigr)\,\exp\Bigl(-\sqrt{n(n+1)}\,\frac{\Delta\tau}{\alpha}\Bigr). \end{aligned} \tag{72}$$

Notice that a difference between two neighbouring energy levels (70) can be estimated. After some appropriate series expansions, which are valid at $\forall n > 1$, one obtains:

$$E_{n+1} - E_n \approx \frac{1}{\alpha}\Big[1 + \frac{1}{8n^2} - \frac{1}{4n^3} + \dots\Big]\,. \tag{73}$$

Equation (73) demonstrates that the levels (70) are approximately equidistant provided the inverse powers of n are neglected in (73) for sufficiently large $n > n_0$. In its turn, the following estimation is also valid:

$$\frac{n+1/2}{\sqrt{n(n+1)}} = 1 + \frac{1}{8n^2} - \frac{1}{8n^3} + \dots\,. \tag{74}$$

It is remarkable that the terms $\propto n^{-1}$ are absent both in (73) and (74). Let us remind that leading asymptotical estimations, which are obtainable with so-called *logarithmic accuracy*, are important for physical applications. The problem at hands just admits an estimation with leading logarithmic accuracy, since inverse powers of n can be omitted with the same (and good) accuracy in (73) and (74) at sufficiently large n. Convergency of the series (72) is not affected in this situation, since $\Delta\tau$ is non-zero.

It is known that at sufficiently large n, the following asymptotics for the Legendre polynomials P_n is valid [21]:

$$P_n(\cos\vartheta) = \sqrt{\frac{2}{\pi n \sin\vartheta}}\,\cos\Big[\big(n+\tfrac{1}{2}\big)\vartheta - \frac{\pi}{4}\Big] + O(n^{-3/2})\,, \quad 0 < \vartheta < \pi\,. \tag{75}$$

Let us split the sum over n in (72) into two parts: $\sum\limits_{n=1}^{n=n_0}$ and $\sum\limits_{n=n_0+1}^{n=\infty}$. Further, let us assume that n_0 is large enough to substitute at $n > n_0$ the Legendre polynomials P_n by their asymptotical expressions given by (75) (and denoted below as $\bar{P}_n(\cos\vartheta)$). Then, using (73) and (74), we can put $G(x,\tau;x',\tau')$ (72) into the following approximate form [6]:

$$\begin{aligned} G(x,\tau;x',\tau') \approx{}& \frac{-g\beta}{4R_c}\left[\left(\frac{1}{2} - \frac{\Delta\tau}{\beta}\right)^2 - \frac{1}{12}\right] - \frac{g}{2\hbar v}\sum_{n=1}^{n_0}\Bigg[\frac{n+\frac{1}{2}}{\sqrt{n(n+1)}}\,P_n\Big(\frac{x}{R_c}\Big)P_n\Big(\frac{x'}{R_c}\Big) \\ &\times \exp\Big(-\sqrt{n(n+1)}\,\frac{\Delta\tau}{\alpha}\Big) - \bar{P}_n\Big(\frac{x}{R_c}\Big)\bar{P}_n\Big(\frac{x'}{R_c}\Big)\exp\Big(-\big(n+\tfrac{1}{2}\big)\,\frac{\Delta\tau}{\alpha}\Big)\Bigg] - \\ &-\frac{g}{2\hbar v}\,\mathrm{e}^{-\Delta\tau/(2\alpha)}\sum_{n=1}^{\infty} t^n\,\bar{P}_n\Big(\frac{x}{R_c}\Big)\bar{P}_n\Big(\frac{x'}{R_c}\Big)\,, \end{aligned} \tag{76}$$

where $t \equiv \exp(-\Delta\tau/\alpha)$, and n_0 is the number, which is fixed (its specific value is forbidden to go to infinity). Expression (76) is valid when τ and τ' are close either to zero or to β. Besides, we assume that $\tau \neq \tau'$ in order to keep convergency of (76), while the term omitted can be estimated [6].

Provided a smallness of x/R_c, x'/R_c and $\Delta\tau/\alpha$ is taken into account, a leading logarithmic behaviour of the series in (76) can be established by the standard tools [21]. Since in the logarithmic approximation the first two terms in (76) are less important in comparison to the third one, we write down the leading contribution to the Green function $G(x,\tau;x',\tau')$ in the quasi-homogeneous limit as follows:

$$G(x,\tau;x',\tau') \simeq \frac{-\Lambda}{2\pi\hbar v}\,\frac{1}{\rho_{\rm TF}(S)}\,\ln\frac{R_c}{\big||x-x'| + \mathrm{i}\hbar v(\tau-\tau')\big|}\,, \tag{77}$$

where $\rho_{\rm TF}$ is given by (23), S is a half-sum of x and x', and it is assumed that

$$u_* \equiv \frac{\big||x-x'| + i\hbar v(\tau-\tau')\big|}{R_c} \ll 1\,. \tag{78}$$

Besides, at sufficiently large n_0, in our consideration it is more appropriate to keep n instead of $n+\frac{1}{2}$ in (75). Expression (77) takes place provided the following conditions of validity of the logarithmic estimation are respected:

$$1 \ll n_0 < \frac{1}{u_*} \ll \frac{1}{u_*}\ln\frac{1}{u_*}\,. \tag{79}$$

A specific value of n_0 can be related to the size of the trap R_c: at a restricted range of deviations between the spatial arguments x and x', increasing of R_c implies increasing of an upper bound for admissible values of n_0. However, due to (79), the estimation obtained for $G(x,\tau;x',\tau')$ (77) does not depend explicitly on a specific choice of n_0. In the limit $1/R_c \to 0$, the total coefficient in front of the logarithm in (77) acquires the value $-1/\theta$, where the critical exponent θ is defined like in (53), (54).

Eventually, we obtain the following estimation for the two-point correlator $\Gamma(x_1,\tau_1;x_2,\tau_2)$:

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \frac{\sqrt{\widetilde{\rho}(x_1)\widetilde{\rho}(x_2)}}{\big||x_1-x_2| + i\hbar v(\tau_1-\tau_2)\big|^{1/\theta(S)}}\,, \tag{80}$$

where the notation for the critical exponent $\theta(S)$ is given by (62).

The estimation obtained (80), where the critical exponent is $\theta(S)$ (62), constitutes the main result of the present subsection devoted to the case given by $k_BT \ll \hbar v/R_c$. From a comparison with the spatially homogeneous Bose gas, one can see that now the derivation of the estimate (80) is just analogous to a transition from the relation (52) to the final asymptotics (53). Then, validity of the corresponding limiting condition $R_c/(\hbar\beta v) \to 0$ means that the result (80) is also due to the fact that the condensate boundary R_c increases slower than the inverse temperature.

Therefore, under the different conditions, $k_BT \gg \hbar v/R_c$ and $k_BT \ll \hbar v/R_c$, we demonstrated in the present section that the behaviour of the correlator in the limit of zero temperature, $(\hbar\beta v)^{-1} \to 0$, and of infinite size of the trap, $1/R_c \to 0$, is given by the coinciding estimations (67) and (80), i.e. the two-point correlation function $\Gamma(x_1,\tau_1;x_2,\tau_2)$ has a unique power-law behaviour in this limit.

5 Conclusion

The model considered describes a spatially non-homogeneous weakly repulsive Bose gas subjected to an external harmonic potential. The functional integral representation for the two-point correlation function is estimated by means of the stationary phase approximation. The main results are obtained for the case when the size of the domain occupied by the quasi-condensate increases, while the temperature of the system goes to zero. In the one-dimensional case, the behaviour of the two-point correlation function at zero temperature is governed by a power law. However, in contrast with the case of spatial homogeneity of the Bose gas, the corresponding critical exponent depends on the same spatial arguments as the correlator itself. It is just the presence of the external potential which is responsible for the non-homogeneity of the critical exponent.

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